

The premetric equivalent of general relativity: teleparallelism

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Abstract

In general relativity (GR), the metric tensor of spacetime is essential since it represents the gravitational potential. In other gauge theories (such as electromagnetism), the so-called premetric approach succeeds in separating the purely topological field equation from the metric-dependent constitutive law. We show here that GR allows for a premetric formulation, too. For this purpose, we apply the teleparallel approach of gravity, which represents GR as a gauge theory based on the translation group. We formulate the metric-free topological field equation and a general linear constitutive law between the basic field variables. The requirement of local Lorentz invariance turns the model into a full equivalent of GR. Our approach opens a way for a natural extension of GR to diverse geometrical structures of spacetime. *file teleGR12.tex*

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1 Introduction

The premetric formalism is an alternative representation of a classical field theory in which the field equations are formulated without the spacetime metric. Only the constitutive relations between the basic field variables, excitation H and field strength F , can involve the metric of the underlying manifold. This idea can be traced back to the early 1920s where it appears in the publications of Kottler [1], [2]. Various applications of this construction

to the formal structure of electrodynamics were worked out by Post [3]. The premetric formalism was studied intensively in the book [4]. For an account of the recent developments in this area, see our review [5].

One advantage of the premetric formalism is that the validity of a certain premetric model can be extended to a more general spacetime geometry. The premetric construction works pretty well in Maxwell’s classical electrodynamics. In this case, all basic ingredients, such as the field equations, the conserved quantities of electric charge and of magnetic flux, and the Lorentz force expression are presented in a metric-free form. Only the constitutive relation between the excitation and the field strength are formulated with the use of the metric tensor. And this relation can be straightforwardly extended to a local and linear relation thereby getting rid of the metric altogether. Let us briefly recall the various outputs of this approach:

- Natural extension of standard electrodynamics by the axion, skewon, and dilaton fields;
- metric-free dispersion relation for electromagnetic waves in general linear response media;
- metric-free Green tensor (photon propagator);
- metric-free jump conditions that include boundary conditions between two media, initial Cauchy conditions, and wave-front conditions;
- Derivation of the metric from the local and linear constitutive relation by prohibiting birefringence in electromagnetic wave propagation;
- natural account of Lorentz violation models.

Although Kottler’s premetric program works well in Newtonian gravity [1] and even in a flat gravitomagnetism model [5], it seems to be completely unacceptable in general relativity (GR). It is due to the well-known fact that Einstein’s theory is essentially based on pseudo-Riemann geometry with the metric tensor as its primary variable. Nevertheless, in this paper, we will show that a premetric construction of GR is possible if one turns to its teleparallel reformulation.

The organization of the paper is as follows: In Section 2, we construct a teleparallel model for the coframe field. It is a vector-valued analog of electromagnetic theory with a well-defined gravitational energy-momentum

current and a Lorentz-type force density. The general local linear constitutive law between the coframe excitation and the coframe field strength is defined by use of a constitutive pseudotensor of 6th rank. In Section 3, we consider the coframe model on a pseudo-Riemannian manifold. This restriction naturally requires the localization of the group of coframe transformations. Moreover, when the constitutive pseudotensor is assumed to be constructed from the metric, the model turns out to be fully equivalent to GR. Section 4 is devoted to the Lorentz force as an interaction term in the equation of motion for a particle. We construct a metric-free equation for a congruence of trajectories with a constitutive law between the momentum covector and the velocity vector. Its restriction to the metric manifold yields a geodesic curve in the gravitational case and the trajectory of a charge in an exterior field in the electromagnetic case. In the concluding section, we discuss the main properties of our construction and propose some possible extensions of standard GR. In the Appendix we provide some technical calculations.

2 Premetric electrodynamics and its coframe analog in gravity

As it is shown in [4], classical electrodynamics can be expressed in a premetric way. In this section, we briefly recall the basic electromagnetic quantities and construct their coframe analogs.

Our key assumption that a gauge field-theoretical model of gravity must be based on a conserved current, here on the macroscopic (“bosonic”) energy-momentum current of matter, see Blagojević et al. [6]. This is in analogy to the electric current that serves as a basis of electromagnetic theory. We use a covector-valued 3-form as a representation of the material energy-momentum current and construct a vector-valued field model. It is presented as a vector-valued analog of the electromagnetic theory, which is given in an ordinary (scalar-valued) differential form formalism. Although at this stage, our construction is only formal, its justification is based on this relation to the energy-momentum conservation law. The existence of an independent conserved untwisted 2-form is naturally related to the definition of a special coframe field on the manifold.

2.1 Geometric structure

Let be given a differential manifold \mathcal{M} endowed with a coframe field ϑ^α . The 1-forms ϑ^α , with $\alpha = 0, 1, 2, 3$, are assumed to be linearly independent at every point of \mathcal{M} .

At this stage, we postulate that all equations are invariant under *rigid linear transformations* of the coframe ϑ^α . The transformed coframe $\vartheta^{\alpha'}$ then becomes

$$\vartheta^{\alpha'} = L_\alpha^{\alpha'} \vartheta^\alpha, \quad L_\alpha^{\alpha'} = \text{const}, \quad (1)$$

with a constant invertible matrix $L_\alpha^{\alpha'} \in GL(4, \mathbb{R})$.

The coframe and its exterior products (taken in increasing order) generate the bases

$$\vartheta^\alpha, \quad \vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta, \quad \vartheta^{\alpha\beta\gamma} := \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma, \quad \vartheta^{\alpha\beta\gamma\delta} := \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta \quad (2)$$

of the spaces of untwisted differential forms of the order 1, 2, 3, and 4, respectively. Under the transformation (1), the basis forms (2) transform as tensors.

In order to express the *twisted forms*, we need the volume element (a non-negative measure) defined on M . Relative to the basis ϑ^α , it is defined as a twisted scalar-valued 4-form

$$\text{vol} = |\vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3| = \frac{1}{4!} |\epsilon_{\alpha\beta\gamma\delta} \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta|. \quad (3)$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ is Levi-Civita's permutation pseudotensor that is normalized as $\epsilon_{0123} = 1$. The absolute value of the 4-form means that in an arbitrary coordinate system this form is represented by the absolute value of a scalar factor multiplied by the positive Riemann (Lebesgue) measure d^4x

$$\text{vol} = |\det(\vartheta_i^\alpha)| d^4x. \quad (4)$$

Under the transformation of the coframe (1), the volume element (3) transforms according to the law

$$\text{vol} \rightarrow |\det L| \text{vol}, \quad (5)$$

with $\det L$ as the Jacobian of the coframe transformation. Thus, the volume element (3) remains positive for all admissible coframes.

The frame field e_a is uniquely defined as the inverse of the coframe,

$$e_\alpha \lrcorner \vartheta^\beta = \vartheta^\beta(e_\alpha) = \delta_\alpha^\beta. \quad (6)$$

Under the coframe transformation (1), the frame obeys the transformation law

$$e_\alpha \rightarrow e_{\alpha'} = (L^{-1})_{\alpha'}{}^\alpha e_\alpha, \quad (7)$$

with $(L^{-1})_{\alpha'}{}^\alpha L_\alpha{}^{\beta'} = \delta_{\alpha'}^{\beta'}$.

With these definitions at hand, the sets

$$\text{vol}, \quad \epsilon_\alpha = e_\alpha \rfloor \text{vol}, \quad \epsilon_{\alpha\beta} = e_\alpha \rfloor \epsilon_\beta, \quad \epsilon_{\alpha\beta\gamma} = e_\alpha \rfloor \epsilon_{\beta\gamma}, \quad (8)$$

with the indices taken in increasing order, serve as basis forms for the spaces of twisted 4-forms, 3-forms, 2-forms and 1-forms, respectively. These basis forms transform with an additional factor $|\det L|$. The two latter forms in (8) are totally antisymmetric.

Since at the first stage, we allow only *global* (rigid) transformations of the coframe, the exterior derivatives of the basis forms (2) and (8) transform as tensors. Hence, one does not need here an exterior covariant derivative of the forms. In other words, we picked a gauge such that the connection Γ vanishes in the frames under consideration: $\Gamma = 0$. We will discuss in the sequel how the symmetry transformation (1) can be generalized to the case of a point dependent $L_\alpha{}^{\alpha'}(x)$.

2.2 Excitation

Electromagnetism: In electromagnetism, the inhomogeneous field equation can be treated as a result of the electric charge conservation law. In order to describe, in a given spatial volume, the electric charge with a prescribed sign, we must use the twisted 3-form J of the electric current. Its expression in a twisted basis reads

$$J = J^\alpha \epsilon_\alpha. \quad (9)$$

Under the coframe transformations (1), we have $\epsilon_\alpha \rightarrow \epsilon_{\alpha'} = |\det L| (L^{-1})_{\alpha'}{}^\alpha \epsilon_\alpha$, and the components of the 3-form J transform as

$$J^\alpha \rightarrow J^{\alpha'} = (\det L)^{-1} L_\alpha{}^{\alpha'} J^\alpha, \quad (10)$$

or

$$J \rightarrow \frac{|\det L|}{\det L} J. \quad (11)$$

The 3-form J remains the same under orientation preserving transformations while picking up an additional sign under the transformations which reverse

the orientation of the coframe. This additional sign compensates the change of the orientation of the integration domain. Consequently, the integral $\int_{\Omega_3} J$ (in particular, the total charge for a closed spatial domain Ω_3) is invariant under the coframe transformations.

The electric charge conservation law in integral and differential forms read, respectively, as

$$\int_{\partial\Omega_4} J = 0, \quad dJ = 0. \quad (12)$$

Locally, the latter relation is equivalent to the inhomogeneous Maxwell equation

$$dH = J, \quad (13)$$

where H is a twisted 2-form of the *electromagnetic excitation*. In the $\vartheta^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ bases, it is expressed as

$$H = \frac{1}{2} H_{\alpha\beta} \vartheta^{\alpha\beta} = \frac{1}{2} \check{H}^{\alpha\beta} \epsilon_{\alpha\beta}, \quad \text{with} \quad \check{H}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} H_{\gamma\delta}. \quad (14)$$

Gravity: Similarly to this electrodynamics construction, we start our gravity model with a conservation law, now it is the energy-momentum conservation. In the canonical formalism, the standard energy-momentum tensor is replaced by an energy-momentum current Σ_α , a covector-valued twisted 3-form. We decompose it with respect to the 3-forms ϵ_β ,

$$\Sigma_\alpha = \Sigma_\alpha{}^\beta \epsilon_\beta. \quad (15)$$

This is an object of 16 independent components, see [7]. Its symmetry may be imposed only by the use of the metric tensor.

Taking into account (1), with constant $L_\beta{}^\alpha$, the conservation law for the energy-momentum current can be expressed as

$$\int_{\partial\Omega_4} \Sigma_\alpha = 0, \quad d\Sigma_\alpha = 0. \quad (16)$$

Using the standard differential-geometry facts we can solve the equation (16) in a small topologically good region as

$$dH_\alpha = \Sigma_\alpha. \quad (17)$$

In this way, a covector valued twisted 2-form of the *gravitational excitation*

$$H_\alpha = \frac{1}{2} H_{\beta\gamma\alpha} \vartheta^{\beta\gamma} = \frac{1}{2} \check{H}^{\beta\gamma}{}_\alpha \epsilon_{\beta\gamma} \quad (18)$$

is defined (up to a total derivative). It is of decisive importance to recognize that there is a fundamental difference to the electromagnetic case (13). The electromagnetic field does not carry electric charge, the gravitational field, however, carries energy-momentum of its own. Hence the right-hand side of (17) reads $\Sigma_\alpha = {}^{(\text{m})}\Sigma_\alpha + {}^{(\vartheta)}\Sigma_\alpha$. Here (m) denotes matter and (ϑ) the coframe field, and we assume additivity of the corresponding energy-momenta.

2.3 Field strength

Electromagnetism: In electrodynamics, the untwisted 2-form of the field strength

$$F = \frac{1}{2} F_{\alpha\beta} \vartheta^{\alpha\beta} \quad (19)$$

satisfies the equations

$$\int_{\partial\Omega_3} F = 0, \quad dF = 0. \quad (20)$$

The homogeneous Maxwell equation $dF = 0$ is an expression of the conservation of the magnetic flux. The electromagnetic field strength F is determined operationally via the Lorentz force density, which acts on the electric current. We will discuss this below. The solution of Eq.(20) is expressed in term of the *electromagnetic potential* A

$$dA = F. \quad (21)$$

In the coframe basis, this untwisted 1-form reads

$$A = A_\alpha \vartheta^\alpha. \quad (22)$$

It is defined up to an addition of a total derivative $A \rightarrow A + d\varphi$.

Gravity: In analogy to the field strength F of the electromagnetic theory, we introduce the *gravitational field strength* F^α . It is an untwisted vector-valued 2-form that satisfies the equation

$$dF^\alpha = 0. \quad (23)$$

The solution of this equation can be locally expressed as

$$F^\alpha = d\theta^\alpha . \quad (24)$$

The set of four 1-forms θ^α is an analog of the electromagnetic potential A . We assume now that the potentials θ^α are linearly independent. It always can be reached due to the gauge invariance of the equation (23). Indeed, we can redefine $\theta^\alpha \rightarrow \theta^\alpha + df^\alpha$, with four arbitrary scalar functions f^α .

We identify the reference coframe ϑ^α with the dynamical coframe θ^α and rewrite (24) as

$$F^\alpha = d\vartheta^\alpha . \quad (25)$$

Thus, we can consider the covector-valued forms Σ_α, H_α and the vector-valued F^α to be related to this special basis. In particular, we express the untwisted form F^α relative to the untwisted basis

$$F^\alpha = \frac{1}{2} F_{\beta\gamma}{}^\alpha \vartheta^{\beta\gamma} . \quad (26)$$

2.4 Lorentz force density

Electromagnetism: In electromagnetism, the Lorentz force density acting on the test particles can be represented by a twisted covector-valued 4-form f_α . Being a top-order form it can be represented as a vector-valued scalar \mathfrak{f}_α multiplied by the volume form $f_\alpha = \mathfrak{f}_\alpha \text{vol}$. In electrodynamics, see [4], the Lorentz force density is given by

$$f_\alpha = (e_\alpha \rfloor F) \wedge J . \quad (27)$$

In the case of a point-wise charge q with a 4-velocity $u = u^\alpha e_\alpha$, the twisted 3-form of the electric current can be expressed as

$$J = qu \rfloor \text{vol} = qu^\alpha \epsilon_\alpha . \quad (28)$$

Consequently, the Lorentz force density (27) reads

$$f_\alpha = (qu^\beta F_{\alpha\beta}) \text{vol} , \quad (29)$$

with the scalar factor represented the standard expression of the Lorentz force covector

$$\mathfrak{f}_\alpha = qu^\beta F_{\alpha\beta} . \quad (30)$$

Gravity: In an analogy to electromagnetism, we assume the Lorentz force density to correspond to the coframe field in the form

$$f_\alpha = (e_\alpha \rfloor F^\beta) \wedge {}^{(\text{m})}\Sigma_\beta. \quad (31)$$

For a point-wise particle of the mass m and the momentum $p = p_\alpha \vartheta^\alpha$, the energy-momentum tensor density is given in the form

$${}^{(\text{m})}\Sigma_\alpha = p_\alpha u \rfloor \text{vol} = p_\alpha u^\beta \epsilon_\beta. \quad (32)$$

Substituting this expression into (31) and using the basis representation (26), we obtain the Lorentz force density in the form

$$f_\alpha = (u^\beta p_\gamma F_{\alpha\beta}{}^\gamma) \text{vol}. \quad (33)$$

The scalar factor of this 4-form represents the covector of the gravitational Lorentz force

$$\mathfrak{f}_a = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma. \quad (34)$$

2.5 Energy-momentum current of gravity

Electromagnetism: The energy-momentum current of electromagnetic field, see [4], is a covector-valued 3-form represented by

$${}^{(\text{em})}\Sigma_\alpha = \frac{1}{2} [F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)]. \quad (35)$$

Alternatively, we can put it into the form

$${}^{(\text{em})}\Sigma_\alpha = e_\alpha \rfloor {}^{(\text{em})}\Lambda + F \wedge (e_\alpha \rfloor H) = -e_\alpha \rfloor {}^{(\text{em})}\Lambda - H \wedge (e_\alpha \rfloor F), \quad (36)$$

with the twisted electromagnetic Lagrangian 4-form

$${}^{(\text{em})}\Lambda := -\frac{1}{2} F \wedge H. \quad (37)$$

Using $F = dA$, we can rederive the field equation $dH = J$ and the current (35) from the Lagrangian (37).

Gravity: Completely similar to the electromagnetic case, we postulate the energy-momentum current for the coframe field as

$${}^{(\vartheta)}\Sigma_\alpha = \frac{1}{2} [F^\beta \wedge (e_\alpha \rfloor H_\beta) - H_\beta \wedge (e_\alpha \rfloor F^\beta)]. \quad (38)$$

We can also introduce the Lagrange 4-form for the coframe field

$${}^{(\vartheta)}\Lambda = -\frac{1}{2}F^\alpha \wedge H_\alpha. \quad (39)$$

Using this expression, we can write the energy-momentum current for the coframe field in a form similar to (36),

$${}^{(\vartheta)}\Sigma_\alpha = e_\alpha \rfloor {}^{(\vartheta)}\Lambda + F^\beta \wedge (e_\alpha \rfloor H_\beta) = -e_\alpha \rfloor {}^{(\vartheta)}\Lambda - H_\beta \wedge (e_\alpha \rfloor F^\beta). \quad (40)$$

2.6 Constitutive relation

In order to complete the electromagnetic and the coframe field models, a *constitutive relation* between the basic variables, namely between excitation H and field strength F should be introduced.

Electromagnetism: The system of the premetric field equations for electromagnetism (13) and (20) involves 8 equations for 12 independent variables, the components of the 2-forms H and F . This system is undetermined and has to be supplemented by an additional relation between the basic variables. In solid state electromagnetism, such relation can be of a rather complicated form. However, even the simplest case of a linear constitutive relation has a wide range of applications.

Using the expansions

$$H = \frac{1}{2}\check{H}^{\alpha\beta}\epsilon_{\alpha\beta}, \quad F = \frac{1}{2}F_{\alpha\beta}\vartheta^{\alpha\beta}, \quad (41)$$

we postulate the most general local linear constitutive relation in the form of

$$\check{H}^{\alpha\beta} = \frac{1}{2}\chi^{\alpha\beta\gamma\delta}F_{\gamma\delta}. \quad (42)$$

Due to this definition, the constitutive pseudo-tensor χ satisfies the symmetry relations

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} = -\chi^{\alpha\beta\delta\gamma}. \quad (43)$$

Gravity: Similarly, our coframe system must be endowed with the constitutive relation between F^α and H_α . We assume this relation to be linear and local. In analogy to electromagnetism, we use

$$H_\alpha = \frac{1}{2}\check{H}^{\beta\gamma}{}_\alpha\epsilon_{\beta\gamma}, \quad F^\alpha = \frac{1}{2}F_{\beta\gamma}{}^\alpha\vartheta^{\beta\gamma}. \quad (44)$$

We postulate the most general local and linear constitutive relation in the form of

$$\check{H}^{\beta\gamma}{}_{\alpha} = \frac{1}{2} \chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} F_{\nu\rho}{}^{\mu}. \quad (45)$$

Here $\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}$ is the constitutive pseudo-tensor that obeys the symmetries

$$\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = -\chi^{\gamma\beta}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = -\chi^{\beta\gamma}{}_{\alpha}{}^{\rho\nu}{}_{\mu}. \quad (46)$$

2.7 Lagrange formalism

In this section, we apply the Lagrange formalism to derive the statements proposed above. This way we are able to justify the coframe model that was postulated in the previous section only by an analogy.

Electromagnetism: Although the electromagnetic case is well-known, it is instructive to recall the variation procedure. This construction turns to be completely metric-free. As only restriction, we will use an additional symmetry relation of the constitutive pseudotensor, namely

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta}. \quad (47)$$

In term of the irreducible decomposition [4], it means that the skewon part is assumed to be forbidden and the constitutive pseudotensor is left with 21 independent components; then and only then is a Lagrange formalism possible.

We start with the Lagrange 4-form

$$\Lambda = -\frac{1}{2} F \wedge H(F) + A \wedge J = \left(-\frac{1}{2} F_{\alpha\beta} \check{H}^{\alpha\beta}{}_{\gamma\delta} (F_{\gamma\delta}) + A_{\alpha} J^{\alpha} \right) \text{vol}. \quad (48)$$

The variation of this Lagrangian takes the form

$$\delta\Lambda = -\frac{1}{2} (\delta F \wedge H + F \wedge \delta H) + \delta A \wedge J. \quad (49)$$

In the case of the linear constitutive relation with the symmetry (47), the first two terms on the right-hand side of Eq.(49) are equal one another. Indeed, using the component representation, we have

$$\begin{aligned} F \wedge \delta H &= -\frac{1}{2} (F_{\alpha\beta} \delta \check{H}^{\alpha\beta}) \text{vol} = -\frac{1}{4} (F_{\alpha\beta} \chi^{\alpha\beta\gamma\delta} \delta F_{\gamma\delta}) \text{vol} \\ &= -\frac{1}{2} (\check{H}^{\gamma\delta} \delta F_{\gamma\delta}) \text{vol} = \delta F \wedge H. \end{aligned} \quad (50)$$

Consequently Eq.(49) takes the form

$$\delta\Lambda = -d(\delta A) \wedge H + \delta A \wedge J = -d(\delta A \wedge H) - \delta A \wedge (dH - J). \quad (51)$$

In order to derive the field equation from this expression, we remove, as usual, the total derivative term and require $\delta\Lambda$ to be zero for arbitrary variations of the potential. Consequently, we obtain the inhomogeneous Maxwell field equation and the electric charge conservation law as its straightforward consequences,

$$dH = J, \quad dJ = 0. \quad (52)$$

Let us now study relation (51) on shell, i.e, we assume that the inhomogeneous Maxwell equation (52) is already satisfied. We are left with

$$\delta\Lambda = -d(\delta A \wedge H). \quad (53)$$

For variations induced by diffeomorphisms, we use $\delta_\alpha\Lambda$ instead of $\delta\Lambda$ and $\delta_\alpha A$ instead of δA . These variations are generated by the Lie derivatives relative to the basis frame vectors, $\delta_\alpha = \mathcal{L}_{e_\alpha}$. We have

$$\delta_\alpha\Lambda = \mathcal{L}_{e_\alpha}\Lambda = d(e_\alpha \rfloor \Lambda), \quad (54)$$

$$\delta_\alpha A = \mathcal{L}_{e_\alpha}A = d(e_\alpha \rfloor A) + e_\alpha \rfloor dA. \quad (55)$$

Substituting into (53), we obtain a conservation law

$$d^{(\text{em})}\Sigma_\alpha = 0, \quad (56)$$

where

$$^{(\text{em})}\Sigma_\alpha = [e_\alpha \rfloor \Lambda + (e_\alpha \rfloor F) \wedge H] - (e_\alpha \rfloor A) \wedge J. \quad (57)$$

On the right-hand side of this equation, we recognize the energy-momentum of the electromagnetic field and the interaction term.

Gravity: Consider a Lagrangian of a system that includes the coframe field and a matter field

$$\Lambda = \frac{1}{2}F^\alpha \wedge H_\alpha + {}^{(\text{m})}\Lambda. \quad (58)$$

Using (44), we rewrite it as

$$\Lambda = \frac{1}{2}(F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha) \text{vol} + {}^{(\text{m})}\Lambda. \quad (59)$$

Variation of this Lagrangian reads (see Appendix)

$$\delta\Lambda = -d(\delta\vartheta^\alpha \wedge H_\alpha) - \delta\vartheta^\alpha \wedge (dH_\alpha - {}^{(\vartheta)}\Sigma_\alpha - {}^{(\text{m})}\Sigma_\alpha), \quad (60)$$

where the energy-momentum current of the coframe field is expressed as

$${}^{(\vartheta)}\Sigma_\alpha = e_\alpha \lrcorner \Lambda + F^\beta \wedge (e_\alpha \lrcorner H_\beta). \quad (61)$$

The matter energy-momentum current ${}^{(\text{m})}\Sigma_\alpha$ is defined via the relation

$$\delta{}^{(\text{m})}\Lambda = \delta\vartheta^\alpha \wedge {}^{(\text{m})}\Sigma_\alpha. \quad (62)$$

For variations of the coframe that vanish on the boundary, we are left with the field equation

$$dH_\alpha = \Sigma_\alpha, \quad (63)$$

where the total energy-momentum current is given as a sum of the coframe current (61) and the matter current defined in (62)

$$\Sigma_\alpha = {}^{(\vartheta)}\Sigma_\alpha + {}^{(\text{m})}\Sigma_\alpha. \quad (64)$$

Note that the conservation law for this quantity, $d\Sigma_\alpha = 0$, follows straightforwardly from field equation (63).

2.8 Premetric electromagnetism-gravity correspondence

Let us now underline the analogy between our premetric coframe model of gravity and the standard electromagnetic theory.

Table I. Premetric electromagnetism-gravity analogy.

Objects and Laws	Electromagnetism	Gravity
Source current	J	Σ_α
Source conserv. law	$dJ = 0$	$d\Sigma_\alpha = 0$
Excitation	H	H_α
inhom. field equation	$dH = J$	$dH_\alpha = {}^{(\vartheta)}\Sigma_\alpha + {}^{(\text{m})}\Sigma_\alpha$
Field strength	F	F^α
hom. field equation	$dF = 0$	$dF^\alpha = 0$
Potential	A	ϑ^α
Potential equation	$dA = F$	$d\vartheta^\alpha = F^\alpha$
Lorentz force density	$f_\alpha = (e_\alpha \lrcorner F) \wedge J$	$f_\alpha = (e_\alpha \lrcorner F^\beta) \wedge {}^{(\text{m})}\Sigma_\beta$
Energy-momentum	$\Sigma_\alpha = e_\alpha \lrcorner \Lambda + F \wedge (e_\alpha \lrcorner H)$	${}^{(\vartheta)}\Sigma_\alpha = e_\alpha \lrcorner \Lambda + F^\beta \wedge (e_\alpha \lrcorner H_\beta)$
Lagrangian	$\Lambda = -(1/2)F \wedge H$	$\Lambda = -(1/2)F^\alpha \wedge H_\alpha$
Constitutive tensor	$\chi^{\alpha\beta\gamma\delta}$	$\chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu$

3 Field-theoretical models on metric manifolds

All the ingredients of the electromagnetic model as well as the coframe model are premetric. Indeed the metric is not involved in these formulas at all. In the current section, we consider these field models on the manifold endowed with a pseudo-Riemannian metric. In the electromagnetic case, this structure allows to describe the case of vacuum electrodynamics. For the coframe field, we are able to reinstate the standard GR via the premetric formalism.

3.1 Coframe field and metric

We consider a manifold \mathcal{M} endowed with a smooth metric g and restrict the coframe field ϑ^a to be orthonormal relative to this metric. Thus, the metric on our manifold can be expressed as

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta, \quad (65)$$

where $g_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric. In other words, we restrict ourselves to the subgroup $O(1, 3)$ of the pseudo-orthonormal transformations of the coframe

$$\vartheta^\alpha \rightarrow \vartheta^{\alpha'} = L_\alpha^{\alpha'} \vartheta^\alpha. \quad (66)$$

Then the metric satisfies the relation

$$g_{\alpha'\beta'} L_\alpha^{\alpha'} L_\beta^{\beta'} = g_{\alpha\beta}. \quad (67)$$

We observe that the metric expression (65) is invariant under a wider class of transformations that depend on a point $x \in \mathcal{M}$ with $L_\alpha^{\alpha'}(x)$, that is, we have *local coframe transformations*.

In local coordinates $\{x^i\}$, we express the coframe and the frame fields respectively as

$$\vartheta^\alpha = \vartheta_i^\alpha dx^i, \quad e_\alpha = e^i_\alpha \partial_i. \quad (68)$$

In these holonomic coordinates, the components of the metric tensor are expressed as

$$g_{ij} = g_{\alpha\beta} \vartheta_i^\alpha \vartheta_j^\beta, \quad g^{ij} = g^{\alpha\beta} e^i_\alpha e^j_\beta. \quad (69)$$

The volume element (3) takes now the form

$$\text{vol} = \sqrt{-g} d^4x = |\det \vartheta_i^\alpha| d^4x, \quad (70)$$

where $g = \det(g_{ij}) = -(\det \vartheta_i^\alpha)^2$. We recognize in this standard expression the twisted 4-form as it is defined in (3).

3.2 Electrodynamics in vacuum

The standard Maxwell-Lorentz electrodynamics is reinstated in the premetric framework provided the constitutive tensor is expressed in terms of the Minkowski metric as follows:

$$\chi^{\alpha\beta\gamma\delta} = \frac{1}{2} \lambda_0 (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}). \quad (71)$$

Here $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$ denotes the vacuum impedance. In the SI system, its value is $\lambda_0 = 1/(377 \Omega)$. Observe that due to the symmetries of the constitutive tensor, (71) is a unique expression that can be constructed from the metric tensor alone.

We expand the field strength 2-form in the coordinate basis

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j \quad (72)$$

and derive from $dF = 0$ the homogeneous Maxwell equation in its standard form

$$\epsilon^{ijkl} \partial_j F_{kl} = 0. \quad (73)$$

If the constitutive tensor (71) is used, also the inhomogeneous field equation $dH = J$ can be rewritten in the standard notation,

$$\partial_j (F^{ij} \sqrt{-g}) = J^i \sqrt{-g}. \quad (74)$$

This results in the conservation law of the electric current,

$$\partial_i (J^i \sqrt{-g}) = 0. \quad (75)$$

The Lorentz force density in a coframe basis reads

$$f_\alpha = e_\alpha^i F_{ik} J^k \sqrt{-g} d^4x. \quad (76)$$

The scalar factor of this 4-form presents the ordinary expression of the Lorentz force covector

$$\mathfrak{f}_i = F_{ik} J^k. \quad (77)$$

3.3 Constitutive pseudotensor of the coframe

We turn now to the gravitational field model. We require the 6th rank constitutive tensor to be expressed via the metric tensor $g_{\alpha\beta}$ only. Due to the symmetries listed in Eq.(46), the most general expression of this type is

$$\begin{aligned} \chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = & \frac{1}{2\kappa} \left\{ \beta_1 g_{\alpha\mu} (g^{\beta\nu} g^{\gamma\rho} - g^{\gamma\nu} g^{\beta\rho}) + \right. \\ & \beta_2 [(g^{\gamma\rho} \delta_{\alpha}^{\beta} - g^{\beta\rho} \delta_{\alpha}^{\gamma}) \delta_{\mu}^{\nu} - (g^{\gamma\nu} \delta_{\alpha}^{\beta} - g^{\beta\nu} \delta_{\alpha}^{\gamma}) \delta_{\mu}^{\rho}] + \\ & \left. \beta_3 [(g^{\gamma\rho} \delta_{\mu}^{\beta} - g^{\beta\rho} \delta_{\mu}^{\gamma}) \delta_{\alpha}^{\nu} - (g^{\gamma\nu} \delta_{\mu}^{\beta} + g^{\beta\nu} \delta_{\mu}^{\gamma}) \delta_{\alpha}^{\rho}] \right\}, \end{aligned} \quad (78)$$

provided we assume the additional “paircom” symmetry

$$\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = \chi^{\nu\rho}{}_{\mu}{}^{\beta\gamma}{}_{\alpha}. \quad (79)$$

Here $\beta_1, \beta_2, \beta_3$ are dimensionless factors, κ is a dimensionful constant.

A remark is in order concerning the dimensions. The coframe and the translational field strength have the dimensions of a length, $[\vartheta^{\alpha}] = [d\vartheta^{\alpha}] = [\ell]$. Analogously, the translational current and the translational excitation have the same dimension of a momentum: $[\Sigma_{\alpha}] = [H_{\alpha}] = [\text{momentum}] = [\frac{m\ell}{t}]$. As a result, $[F^{\alpha} \wedge H_{\alpha}] = [\frac{m\ell^2}{t}] = [\hbar]$. Hence the Lagrangian has indeed the correct dimension of an action. Consequently, the dimension of the constant κ is obtained as the ratio of the dimension of $[F^{\alpha}]$ divided by the dimension of $[H_{\alpha}]$, that is, we have $[\kappa] = [\frac{t}{m}]$. One can straightforwardly check that $[\kappa c] = [\frac{t}{m}] = [\kappa]$. Thus we conclude that κ can be identified with κc . This demonstrates a remarkable consistency of the teleparallel gravity with Einstein’s general relativity theory.

Observe that the symmetry (79) allows the coframe model to be derived from a Lagrangian. Using the constitutive tensor (78), we write the coframe Lagrangian in (58) as

$$\begin{aligned} {}^{(\vartheta)}\Lambda &= \frac{1}{2} F^{\alpha} \wedge H_{\alpha} = \frac{1}{4\kappa} F_{\beta\gamma\alpha} (\beta_1 F^{\beta\gamma\alpha} + \beta_2 g^{\alpha\beta} F_{\nu}{}^{\gamma\nu} + \beta_3 F^{\alpha\gamma\beta}) \text{vol} \\ &= \frac{1}{2\kappa} F_{\beta\gamma\alpha} (\alpha_1 {}^{(1)}F^{\beta\gamma\alpha} + \alpha_2 {}^{(2)}F^{\beta\gamma\alpha} + \alpha_3 {}^{(3)}F^{\beta\gamma\alpha}) \text{vol}, \end{aligned} \quad (80)$$

where ${}^{(I)}F^{\beta\gamma\alpha}$ are the 3 irreducible pieces of the field strength, see [7].

3.4 GR in terms of coframe variables

We constructed a set of coframe models parametrized by dimensionless numerical parameters α_1, α_2 , and α_3 that turns out to be very similar to the electrodynamics system. The question is: How are these models connected to gravity, in particular to GR?

Recall that Einsteins theory is expressed by the field equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij}. \quad (81)$$

Here $\kappa = 8\pi G/c^4$, with Newton's gravitational constant G . When the metric tensor (69) are substituted in the left-hand side of (81), we obtain the expression that includes the second order derivatives of the coframe components plus the product of their first order derivatives. Exactly the same type of expression we have in the coframe field equation $dH_\alpha = \Sigma_\alpha$. Thus, for some special values of the parameters α_1, α_2 , and α_3 , we can reinstate standard GR from this coframe field equation. It seems technically simpler to deal with the Lagrangian. Recall that the left-hand side of (81) is derived from the action functional

$$\mathcal{A} = \frac{1}{2\kappa c} \int R\sqrt{-g} dx^4. \quad (82)$$

As it is well known, the scalar curvature and in turn the Lagrangian in (82) can be expressed as a sum of two parts: a term that is quadratic in the first order derivatives of the metric plus a total divergence. In particular, up to a total derivative, (82) can be represented, see [8] Eq.(3.20), as

$$\mathcal{A} = \frac{1}{2\kappa c} \int g^{ij} (\Gamma_{li}^k \Gamma_{kj}^l - \Gamma_{lk}^k \Gamma_{ij}^l) \sqrt{-g} dx^4. \quad (83)$$

The expression of this Lagrangian in terms of the coframe is well-known. In a compact form, see [9], this *teleparallel equivalent of GR* reads as follows:

$$\mathcal{A} = \frac{1}{2} \int F^\alpha \wedge H_\alpha, \quad (84)$$

where

$$\begin{aligned} H_\alpha &= \frac{1}{\kappa c} \star [g_{\alpha\beta} F^\beta - g_{\alpha\beta} \vartheta^\beta \wedge (e_\gamma \rfloor F^\gamma) - 2g_{\beta\gamma} e_\alpha \rfloor (\vartheta^\beta \wedge F^\gamma)] \\ &= \frac{1}{\kappa c} g_{\alpha\beta} \star (-^{(1)}F^\beta + 2^{(2)}F^\beta + \frac{1}{2}^{(3)}F^\beta). \end{aligned} \quad (85)$$

In tensor form this formula can be found in [10], see Eq.(A.15).

There is a long development of this teleparallel theory of gravity. Relevant papers are, amongst others, Pellegrini & Plebanski [11], Kaempffer [12], Cho [13], Hehl, Nitsch & von der Heyde [10], Nitsch et al. [14], Muench et al. [15], Nester et al. [16, 17, 18], Obukhov & Pereira [19], Itin [20, 21, 22, 23, 24], Maluf [25], Aldrovandi & Pereira [26]. A review can be found in [6].

We substitute the expression (85) into (84) and compare the result to the coframe Lagrangian (80). Consequently we derive the values of the free parameters as

$$(\beta_1 = 1, \beta_2 = -4, \beta_3 = 2) \quad \text{and} \quad (\alpha_1 = -1, \alpha_2 = 2, \alpha_3 = \frac{1}{2}). \quad (86)$$

Since (39) includes all possible Lagrangians that are quadratic in the first order derivatives of the coframe components, we derive that Hilbert Lagrangian is a special case of the coframe Lagrangian.

3.5 Local coframe transformations

In the premetric teleparallel formalism, GR turns out to be a special case of a general coframe model with the specific parameters of (86). This case, however, is very distinguished. Indeed, the standard GR and its teleparallel equivalent are invariant under *local* Lorentz transformations of the coframe field

$$\vartheta^{\alpha'} = L_{\alpha}^{\alpha'}(x) \vartheta^{\alpha}. \quad (87)$$

It can be checked that there is only one set of the free parameters $(\alpha_1, \alpha_2, \alpha_3)$, which constitutes a locally Lorentz invariant coframe model with invariant Lagrangian and field equation. Other ingredients of the field model, such as field strength, excitation, energy-momentum current and Lorentz-type force, are not local invariants. This fact is very well known in GR, where the energy-momentum of gravity cannot be defined in a covariant way.

4 Lorentz force and geodesics

Let us formulate the premetric equation of motion of a particle in an external gravitational field. We start with a relativistic version of Newton's equation of motion with the gravitational Lorentz force on its right-hand side. Subsequently, we demonstrate that it can be rewritten as the standard

geodesic equation of GR, provided we assume the metric dependent constitutive relation between the particle momentum p_α and the particle velocity u^α .

4.1 Premetric equation of particle motion

In (33), we presented the Lorentz force density as a covector valued twisted 4-form. If a volume factor is extracted from this top form, we are left with the ordinary (untwisted) covector of the Lorentz force

$$\mathfrak{f}_\alpha = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma. \quad (88)$$

Unlike the electromagnetic Lorentz force, this expression includes the product of two variables characterizing the spacetime trajectory of the test particle—the momentum p_α and the velocity u^α . The coframe components of the velocity are represented by the derivative of the trajectory taken with respect to the length parameter s

$$u^\alpha = \vartheta_i{}^\alpha u^i = \vartheta_i{}^\alpha \frac{dx^i}{ds}. \quad (89)$$

In canonical field theory, the momentum is represented by a covector. This is due to the fact that in a Lagrangian the momentum is dual to the velocity.

In order to express the equation of motion of a particle, the force must be equated to the time derivative of the momentum

$$\frac{dp_\alpha}{ds} = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma. \quad (90)$$

We observe that, due to (89), this equation is invariant under arbitrary smooth reparametrization of the curve $s \rightarrow \lambda(s)$. Thus, even being expressed via the length parameter s , Eq.(90) is premetric, provided it is considered as an equation for two independent variables, the momentum p_α and the velocity u^α . Moreover, Eq.(90) is invariant under rescaling of the momentum $p_\alpha \rightarrow C p_\alpha$. This simple symmetry represents Einstein's principle of equivalence of inertial and gravitational mass, which turns out to be valid even in our premetric framework.

Let us rewrite Eq.(90) in a coordinate basis. Multiplying both sides of this equation by $\vartheta_i{}^\alpha$, we find

$$\vartheta_i{}^\alpha \frac{dp_\alpha}{ds} = u^j (F_{\alpha\beta}{}^\gamma \vartheta_i{}^\alpha \vartheta_j{}^\beta) p_\gamma. \quad (91)$$

Evaluating $F_{\alpha\beta}{}^\gamma$ in a coordinate basis, we obtain

$$\vartheta_i^\alpha \frac{dp_\alpha}{ds} = u^j (\partial_i \vartheta_j^\gamma - \partial_j \vartheta_i^\gamma) p_\gamma, \quad (92)$$

or, equivalently,

$$\vartheta_i^\alpha \frac{dp_\alpha}{ds} + \frac{d\vartheta_i^\alpha}{ds} p_\alpha = u^j p_\gamma \partial_i \vartheta_j^\gamma. \quad (93)$$

Consequently, the equation of motion of a particle takes the form

$$\frac{dp_i}{ds} = u^j p_\alpha \partial_i \vartheta_j^\alpha. \quad (94)$$

Since this equation is metric-free, it is valid in a general geometric background.

4.2 Geodesic equation

Eventually, we endow the manifold with a metric tensor g . Recall the two equivalent representations of the metric tensor in terms of a coframe ϑ^α or of coordinates x^i , respectively:

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta = g_{ij} dx^i \otimes dx^j. \quad (95)$$

We observe

$$p_\gamma \partial_i \vartheta_j^\gamma = g_{\beta\gamma} p^k \vartheta_k^\beta \partial_i \vartheta_j^\gamma = \frac{1}{2} p^k \partial_i g_{jk}. \quad (96)$$

Consequently, (94) takes the form

$$\frac{dp_i}{ds} = u^j \partial_j p_i = \frac{1}{2} \partial_i g_{jk} p^j u^k. \quad (97)$$

This equation includes two unknowns, the covector p_i and the vector u^i . We assume the *constitutive relation* between the momentum and the velocity of the particle to be local and linear,

$$p_i = m g_{ij} u^j, \quad (98)$$

where m is the mass of the particle. Accordingly, we have the standard relation $p_i = m u_i$. Thus, (97) takes the form

$$\frac{du_i}{ds} = u^j \partial_j u_i = \frac{1}{2} \partial_i g_{jk} u^k u^j. \quad (99)$$

This is equivalent to the standard geodesic equation, see [27]:

$$\frac{du^i}{ds} + \Gamma_{jk}^i u^j u^k = 0 \quad (100)$$

4.3 Particle motion in an electromagnetic field

Having at our disposal the premetric equation, which results in a geodesic, we are now able to postulate the equation describing the motion of a point-wise electric charge. The total acceleration term must be the sum of the gravitational and the electromagnetic Lorentz terms

$$\frac{dp_\alpha}{ds} = {}^{(\vartheta)}\mathfrak{f}_\alpha + {}^{(\text{em})}\mathfrak{f}_\alpha. \quad (101)$$

We use the electromagnetic Lorentz force given in (30) and the gravitational Lorentz force given in (34) and obtain

$$\frac{dp_\alpha}{ds} = u^\beta p_\gamma F_{\alpha\beta}{}^\gamma + qu^\beta F_{\alpha\beta}. \quad (102)$$

Using the constitutive relation (98), we are end up with the standard equation of motion of a charge in a curved spacetime:

$$\frac{du^i}{ds} + \Gamma_{jk}{}^i u^j u^k = \frac{q}{m} (F^{ij} u_j). \quad (103)$$

5 Discussion

A gauge view at gravity

A gauge-theoretical understanding of gravitational theory was our tool for arriving at a premetric version of general relativity, namely teleparallelism, here specifically by considering a gauge theory of the *translation group*. However, it is the semidirect product of the translation group $T(4)$ with the *Lorentz group* $SO(1,3)$, the Poincaré group $T(4) \rtimes SO(1,3)$, which is the group of motion in Minkowski spacetime. The Poincaré group is connected with the energy-momentum and spin angular momentum of matter as Noether currents.

The gauging, that is, the localization of the Poincaré group, yields the Poincaré gauge theory of gravity (PG), see the review [6], Part B. If the spin of matter is suppressed, a (Inönü-Wigner type) group contraction of the PG leads to a translation gauge theory. This contraction is mathematically very delicate and is conventionally done in a heuristic manner. In this way, the teleparallelism theory is emerging. At the same time it becomes intelligible why teleparallelism has a number of unexpected and somewhat strange

features. After all, the vanishing of the curvature, that is, the defining characteristics of teleparallelism theory, is hard to digest from a purely Einsteinian GR point of view (as already Pauli remarked to Einstein in the 1920s). From the point of view of PG it is self-evident, since the curvature is the gauge field strength belonging to the Lorentz group—and the suppression of the material spin in turn suppresses the Lorentz group as gauge group. And thus the Pauli objection can be invalidated. By the same token we recognize that teleparallelism can only be really understood in the context of PG. It is not comprehensible as a stand-alone theory.

Nonlinear extension of teleparallelism

A further success of the gauge-theoretical view at GR can be listed: When, in the early 2000s, Mashhoon recognized that Einstein’s clock hypothesis is not sustainable as soon as high translational and rotational accelerations occur. Therefore, he looked for a classical nonlocal extension of GR and of the Einstein field equation. In spite of several attempts, he was not able to implement it on the basis of the Einstein equation and GR.

Again, as soon as one looked at gravity from a gauge-theoretical perspective, it is evident of how one has to proceed: Switch from GR to the teleparallel approach to gravity. Its structure is closely related to electromagnetism. And in electromagnetism it is straightforward to generalize a local and linear constitutive law to a *nonlocal* and linear framework—already Volterra pointed this out in the 1910s.

Mashhoon and one of the present authors [28, 29] took their “teleparallel” glasses and looked at the field equation of gravity. Following Volterra, they set up a nonlocal framework for a classical theory of gravity, extending thereby GR to the domain of high accelerations. This nonlocal theory of gravity was worked out in some detail by Mashhoon and collaborators and can be found in the forthcoming monograph of Mashhoon [30]. Quite unexpectedly, nonlocal gravity is able to describe the cosmos without taking recourse to dark matter, see the title of [28]. The nonlocal theory explains dark matter straightforwardly. Up to now, the astrophysical data seem to speak in favor of this new framework.

$U(1)$ -axion field versus axial torsion vector field

Considering axion electrodynamics [31], the $U(1)$ -axion a is present in the 3rd irreducible piece of the electromagnetic constitutive tensor in (42):

$${}^{(3)}\chi^{\alpha\beta\gamma\delta} = a\epsilon^{\alpha\beta\gamma\delta}, \quad [{}^{(3)}\chi^{\alpha\beta\gamma\delta}] = 1/(\text{electric resistance}). \quad (104)$$

Similarly, the axial torsion piece $\mathcal{A} := g_{\alpha\beta}^*(\vartheta^\alpha \wedge F^\beta)$ is manifest in the 3rd piece of the gravitational constitutive tensor in (45):

$${}^{(3)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}, \quad [\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}] = \text{mass/time} = [1/\kappa c]. \quad (105)$$

The explicit form of ${}^{(3)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}$ can be read off most conveniently from the Lagrangian (80). Both pseudoquantities, axion and axial torsion, should contribute to the axial anomaly of quantum field theory, see Obukhov [32].

Moreover, Mielke et al. [33] tentatively assumed that the axial torsion \mathcal{A} , which is a geometric quantity characterizing spacetime, can be chosen as the gradient of a pseudoscalar field \mathcal{P} , that is, $\mathcal{A} = d\mathcal{P}$. Subsequently, without any physical argument to support it and without an appropriate dimensional analysis, \mathcal{P} is identified with the axion field a of the internal $U(1)$ -symmetry of Peccei-Quinn. This is what we call an ad hoc assumption. Moreover, our dimensional analysis in Eqs.(104) and (105) above shows how far-fetched such an assumption is.

Similar attempts were made by Castillo-Felisola et al. [34]. Corral argued that they don't consider torsion as a field strength related to translational gauging, but rather that they rely on "the geometrical interpretation of *torsion*." And this would make a difference. We cannot share this optimism: What else than a geometric quantity is a translational gauge field strength, after all?

One could try the ansatz

$$\widehat{\chi}^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = {}^{(\vartheta)}\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} + a'\varepsilon^{\beta\gamma\nu\rho}g_{\alpha\mu} \quad (106)$$

in order to link (104) with (105). However, the trace via $g^{\alpha\mu}$ of (106) can never yield the axion, unless one introduces in an ad hoc fashion a dimensionful factor in a' . In other words, in this way one cannot find an axion in a natural way.

The $U(1)$ -axion is related to the *internal* group $U(1)$, whereas the axial torsion is related to the *external* translation group $T(4)$ via the Cartan circuit.

One should not marry internal and external groups, unless one investigates supersymmetry, which allows such a mixing.

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A Variation of the coframe Lagrangian

We start with the premetric coframe Lagrangian

$$\Lambda = \frac{1}{2} F^\alpha \wedge H_\alpha(F^\beta). \quad (107)$$

Substituting the component representations of the forms (18,26), we obtain

$$\Lambda = \frac{1}{8} (F_{\beta\gamma}{}^\alpha \check{H}^{\mu\nu}{}_\alpha) \vartheta^{\beta\gamma} \wedge \epsilon_{\mu\nu}. \quad (108)$$

Applying the relation

$$\vartheta^{\beta\gamma} \wedge \epsilon_{\mu\nu} = \delta_\mu^\gamma \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\gamma \quad (109)$$

we derive the coframe Lagrangian in terms of components,

$$\Lambda = \frac{1}{4} (F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha) \text{vol}. \quad (110)$$

Consequently the variation of the Lagrangian takes the form

$$\delta\Lambda = \frac{1}{4} [\delta(F_{\beta\gamma}{}^\alpha) \check{H}^{\beta\gamma}{}_\alpha + F_{\beta\gamma}{}^\alpha \delta(\check{H}^{\beta\gamma}{}_\alpha)] \text{vol} + \frac{1}{4} (F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha) \delta(\text{vol}). \quad (111)$$

Using the linear constitutive relation (45) with its symmetry property (79), we find

$$\begin{aligned} F_{\beta\gamma}{}^\alpha \delta(\check{H}^{\beta\gamma}{}_\alpha) &= F_{\beta\gamma}{}^\alpha \chi^{\beta\gamma}{}_\alpha{}^{\nu\rho}{}_\mu \delta(F_{\nu\rho}{}^\mu) \\ &= \delta(F_{\nu\rho}{}^\mu) \chi^{\nu\rho}{}_\mu{}^{\beta\gamma}{}_\alpha F_{\beta\gamma}{}^\alpha = \delta(F_{\nu\rho}{}^\mu) \check{H}^{\nu\rho}{}_\mu. \end{aligned} \quad (112)$$

Consequently (111) takes the form

$$\delta\Lambda = \frac{1}{2} \delta(F_{\beta\gamma}{}^\alpha) \check{H}^{\beta\gamma}{}_\alpha \text{vol} + \frac{1}{4} F_{\beta\gamma}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha \delta(\text{vol}). \quad (113)$$

In order to calculate the variation $\delta(F_{\beta\gamma}{}^\alpha)$, we use the expression (26):

$$\delta(d\vartheta^\alpha) = \frac{1}{2}\delta(F_{\beta\gamma}{}^\alpha)\vartheta^{\beta\gamma} + F_{\beta\gamma}{}^\alpha\delta\vartheta^\beta \wedge \vartheta^\gamma. \quad (114)$$

Hence,

$$\delta(F_{\beta\gamma}{}^\alpha) = e_\gamma \rfloor e_\beta \rfloor d(\delta\vartheta^\alpha) - F_{\beta\mu}{}^\alpha e_\gamma \rfloor (\delta\vartheta^\mu) + F_{\gamma\mu}{}^\alpha e_\beta \rfloor (\delta\vartheta^\mu). \quad (115)$$

Thus, the first term of (113) reads

$$\frac{1}{2}\delta(F_{\beta\gamma}{}^\alpha)\check{H}^{\beta\gamma}{}_\alpha \text{vol} = -d(\delta\vartheta^\alpha) \wedge H_\alpha - F_{\beta\mu}{}^\alpha \check{H}^{\beta\gamma}{}_\alpha \delta\vartheta^\mu \wedge (e_\gamma \rfloor \text{vol}). \quad (116)$$

In order to calculate the variation of the volume element, we apply the formula

$$\delta(\text{vol}) = \delta\vartheta^\mu \wedge (e_\mu \rfloor \text{vol}). \quad (117)$$

Accordingly, the variation of the coframe Lagrangian (113) takes the form

$$\delta\Lambda = -d(\delta\vartheta^\alpha) \wedge H_\alpha - \Sigma_\alpha \wedge \delta\vartheta^\alpha, \quad (118)$$

where

$$\Sigma_\alpha = \left(F_{\beta\alpha}{}^\mu \check{H}^{\beta\rho}{}_\mu - \frac{1}{4}\delta_\alpha^\rho F_{\beta\gamma}{}^\mu \check{H}^{\beta\gamma}{}_\mu \right) (e_\rho \rfloor \text{vol}). \quad (119)$$

Using the component version of the forms (18,26), we obtain the expressions (38) and (40). We extract the total derivative in (118) and obtain finally

$$\delta\Lambda = -d(\delta\vartheta^\alpha \wedge H_\alpha) - \delta\vartheta^\alpha \wedge (dH_\alpha - \Sigma_\alpha). \quad (120)$$

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